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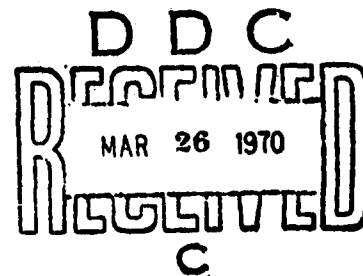
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General Transducer Array Analysis

by

Charles H. Sherman



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Abstract

A mathematical model for transducer arrays is developed which is not restricted to fixed velocity distribution transducers. The model is general enough to cover most transducers of interest for Sonar applications. Its formulation is also explicit and simple enough to make it readily usable in transducer and array design. It is especially well suited for analyzing the effects of transducer head flexing. The use of the model is illustrated by some discussion of a spherical array of longitudinal vibrator transducers with rectangular heads. Numerical calculations are given here for one transducer in air, and for this case some results related to head flexing are obtained and discussed.

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Introduction

Mathematical models for transducer array analysis have usually been based on the assumption of fixed velocity distribution transducers, that is, transducers which vibrate with the same velocity distribution under all conditions.¹ This idealization is often an adequate approximation for underwater sound transducers, but there are important cases where the velocity distribution of a transducer depends significantly on the medium in which it is immersed or on circumstances such as its location in an array. To handle such cases we will develop here a mathematical model without the fixed velocity distribution assumption, which is general enough to cover most transducers and explicit enough to be readily usable in transducer and array design. The model is based on classical modal analysis as used by Lax² for one circular plate with clamped edges in an infinite, plane, rigid baffle.

We will develop the general equations of the model in the first part of this report. They can be used to take head flexing into consideration in transducer and array design and to interpret head flexing measurements such as those which are now being made by holographic interferometry.³ In the second part we will illustrate how to use the model by discussing a spherical array and the common longitudinal resonator type of transducer. We will do numerical calculations for one transducer in air and obtain some results related to head flexing.

It must be remembered that a general model is merely a framework within which more specialized information can be used in a systematic way.

Such a model for transducer arrays does not reduce the need for solutions to all the specific elastic, acoustic and electromechanical problems which are involved. We have tried to formulate the general model in such a way that existing solutions and information about specialized problems can be easily used. It is hoped that this will make the model readily applicable to practical problems.

The Mathematical Model

General Formulation

Consider an arbitrary array in which the radiating surface of each transducer is an elastic body such as a membrane or plate which we will refer to hereafter as the head. One side of each head is in contact with an acoustic fluid medium, while the other side is isolated from the fluid medium but is in contact with the other parts of the transducer such as the electromechanical drive mechanism, tie rod and edge seals. Thus, acoustic forces act on one side of the head from all the transducers in the array and from sources outside the array. Acting on the other side are forces from the electromechanical driver plus constraints exerted by the various connections with the rest of the transducer when the head moves.

The equation of motion for harmonic vibrations of the j th head in an array of N transducers can be written

$$\left[\frac{D_j}{i\omega} + i\omega m(\vec{r}_j) \right] v(\vec{r}_j) = f(\vec{r}_j) - p(\vec{r}_j) \quad , \quad j = 1, 2, \dots, N \quad (1)$$

where

\vec{r}_j = the position vector of a point on the jth head

D_j = a linear differential operator, the form of which depends on the type of head

ω = angular frequency

$m(\vec{r}_j)$ = mass per unit area of the head

$v(\vec{r}_j)$ = normal velocity of the head

$f(\vec{r}_j)$ = normal force per unit area exerted on the head by other parts of the transducer

$p(\vec{r}_j)$ = acoustic pressure on the head

Some specific examples of D are:

$$D = \frac{\gamma t^3}{12(1 - \sigma^2)} \nabla^4 \quad \text{for an elastic plate} \quad (2a)$$

γ = Young's modulus

σ = Poisson's ratio

t = Thickness

$$D = T \nabla^2 \quad \text{for a membrane} \quad (2b)$$

T = Tension per unit length

$$D = Y \tau K^2 \frac{d^4}{dx^4} \quad \text{for an elastic bar} \quad (2c)$$

K = radius of gyration of cross section

(= $\tau/\sqrt{12}$ for a bar of rectangular cross section)

These three examples are the simplest forms of the differential operators for the given elastic bodies. More general operators can also be used; for example⁴, if shear and rotatory inertia are included for an elastic bar we have for harmonic vibrations

$$D = Y \tau K^2 \frac{d^4}{dx^4} + \omega^2 \left(\rho \tau K^2 + \frac{Y \rho \tau K^2}{k' \mu} \right) \frac{d^2}{dx^2} + \omega^4 \rho \tau K^2 \frac{1}{k' \mu} \quad (2d)$$

k' = dimensionless numerical factor (= 2/3 for rectangular cross section)

μ = shear modulus

ρ = density

We can write the acoustic pressure in the form

$$p(\vec{r}_j) = \sum_{i=1}^N \iint v(\vec{r}_i) G(\vec{r}_j, \vec{r}_i) dS_i + p_r(\vec{r}_j), \quad (3)$$

where each term of the sum is the contribution from one of the transducers in the array. G , the appropriate Green's function, is a convenient representation for the present, but the formulation is not restricted to Green's functions. We will soon express the acoustic

terms in such a way that they can be evaluated by any method for calculating sound fields which suits the specific situation. The term $p_r(\vec{r}_j)$ represents sound waves from sources outside the array including diffraction from the transducers and baffle of the array. This term does not depend on the velocity distribution of any of the transducers in the array.

Similarly, we can separate the forces exerted on each head by the rest of the transducer into a part which depends on the transducer velocity distribution and a part which does not:

$$f(\vec{r}_j) = n(\vec{r}_j) E_j - \iint v(\vec{r}_j') z_j(\vec{r}_j, \vec{r}_j') dS_j' . \quad (4)$$

The first term is the blocked force per unit area where $n(\vec{r}_j)$ is the electromechanical transfer function, and E_j is the voltage at the terminals of the electromechanical element of the jth transducer. The second term accounts for all the mechanical constraints between the head and the other parts of the transducer, where $z_j(\vec{r}_j, \vec{r}_j')$ is an acoustic transfer impedance function.

When Eqs.(3) and (4) are combined with Eq.(1) we have

$$p_r(\vec{r}_j) = n(\vec{r}_j) E_j - \left[\frac{\rho_j}{i\omega} + i\omega m(\vec{r}_j) \right] v(\vec{r}_j) - \iint v(\vec{r}_j') z_j(\vec{r}_j, \vec{r}_j') dS_j' - \sum_{i=1}^N \iint v(\vec{r}_i') G(\vec{r}_j, \vec{r}_i') dS_i' \quad (5)$$

The other member of the usual pair of transducer equations expresses the current in terms of the voltage and the velocity. For reciprocal transducers it is

$$I_j = Y_{b,j} E_j + \iint v(\vec{r}_j') n(\vec{r}_j') dS_j' \quad (6)$$

where $Y_{b,j}$ is the blocked electrical admittance. These equations correspond to the general transducer equations of Foldy and Primakoff⁵, except that they use velocity and current as the independent variables. We have also specified the nature of the transducer to a somewhat greater extent, and we are treating arrays rather than a single transducer. Note that the extension of the general transducer equations to arrays has recently been discussed by Hickman, Martin and Schenck.⁶ Hickman⁷ has also given a general mathematical model for sonar transducer array systems which includes amplifier-delay line networks and domes.

In the usual modal analysis⁸ all the terms in Eq.(5) which depend on $\psi(\vec{r}_j)$ would define the eigenvalue problem to be solved. In the present case this set of coupled integro-differential equations is too difficult to be solved directly. Instead we start with the relatively simple eigenvalue problem formed by omitting the integrals from Eq.(5). The solutions of this problem, which we'll call the normal modes of the head and denote by $\eta_n(\vec{r}_j)$, satisfy

$$D\eta_n(\vec{r}_j) = \omega_{nj}^2 \eta_n(\vec{r}_j) \quad (7)$$

plus boundary conditions. For convenience, we will let the single subscript n represent the double subscripts which would be required for two dimensional elastic bodies. This problem, although it consists only of the simplest part of the whole problem being considered, still has known solutions for only a few elastic bodies with simple shapes and simple boundary conditions. Fortunately, many of these are useful for practical transducers. The ω_{nj} in Eq.(7) are the natural frequencies of the modes

of the head. These modes form a complete set of orthogonal functions which we'll normalize such that

$$\iint m(\vec{r}_j) \eta_n(\vec{r}_j) \eta_m(\vec{r}_j) dS_j = M_j \delta_{nm} , \quad (8)$$

where M_j is the total mass of the head. Note that the $\eta_n(\vec{r}_j)$ are then dimensionless.

The normal modes of the head can be used as the basis for expanding the general solution of Eq.(5). Thus we write

$$v(\vec{r}_j) = \sum_{n=0}^{\infty} V_{nj} \eta_n(\vec{r}_j) , \quad (9)$$

where the mode velocities, V_{nj} , are to be determined. We now substitute Eq.(9) into Eq.(5), multiply by $\eta_m(\vec{r}_j)$, integrate over the surface of the jth head, and use Eqs.(7) and (8) to obtain

$$\begin{aligned} \left(i\omega M_j - i \frac{M_j \omega_m^2}{\omega} \right) V_{mj} = & E_j \iint n(\vec{r}_j) \eta_m(\vec{r}_j) dS_j - \sum_n V_{nj} \iiint \eta_n(\vec{r}_j) g(\vec{r}_j, \vec{r}_j') \eta_m(\vec{r}_j) dS_j dS_j' \\ & - \iint p_r(\vec{r}_j) \eta_m(\vec{r}_j) dS_j - \sum_i \sum_n V_{ni} \iiint \eta_n(\vec{r}_i) G(\vec{r}_j, \vec{r}_i) \eta_m(\vec{r}_j) dS_i dS_j . \end{aligned} \quad (10)$$

It will be understood that sums over n or m include all the modes, while sums over i or j include all the transducers in the array. All the integrals on the right of Eq.(10) are calculable from knowledge of the transducer mechanism and construction, the array geometry, and the operating conditions. This set of algebraic equations can be solved for the V_{mj} after truncating to a finite number of modes. If we use N' modes we have NN' equations to solve. The modes of each head are coupled mechanically and acoustically by the second and fourth terms on

the right of Eq.(10). The fourth term also couples the modes of each head to the modes of every other head in the array.

We now define the modal pressure function,

$$p_{ni}(\vec{r}_j) = \iint \eta_n(\vec{r}_i) G(\vec{r}_j, \vec{r}_i) dS_i, \quad (11)$$

which is the sound pressure produced at the point \vec{r}_j per unit velocity amplitude of the nth mode of the ith head. Using Eq.(11) we can write the last integral on the right of Eq.(10) as

$$Z_{nmij} = \iiint \eta_n(\vec{r}_i) G(\vec{r}_j, \vec{r}_i) \eta_m(\vec{r}_j) dS_i dS_j = \iint p_{ni}(\vec{r}_j) \eta_m(\vec{r}_j) dS_j, \quad (12)$$

$$= \frac{1}{V_{ni} V_{mj}^*} \iint V_{ni} p_{ni}(\vec{r}_j) V_{mj}^* \eta_m(\vec{r}_j) dS_j,$$

which is the mutual radiation impedance between the nth mode of the ith head and the mth of the jth head. From the acoustic reciprocity theorem we find

$$Z_{nmij} = Z_{mnji}. \quad (13)$$

Further, since p_{ni} is symmetric (antisymmetric) if η_n is symmetric (antisymmetric), we also have

$$Z_{nmii} = 0 \quad (14)$$

when n and m refer to different symmetries. The quantity p_{ni} is expressed in terms of Green's functions in Eq.(11) only for analytical convenience. It may be calculated by any method available, and the results can then be used in Eq.(12) to calculate the mutual radiation

impedances and later in Eq.(20) to calculate the sound pressure in the near or far field.

It is also convenient to define

$$z_{nmj} = \iiint \eta_n(\vec{r}_j') z(\vec{r}_j, \vec{r}_j') \eta_m(\vec{r}_j) dS_j' dS_j = z_{mni} , \quad (15)$$

$$n_{mj} = \iint \eta(\vec{r}_j) \eta_m(\vec{r}_j) dS_j , \quad (16)$$

$$f_{mj} = \iint p_r(\vec{r}_j) \eta_m(\vec{r}_j) dS_j . \quad (17)$$

Then Eqs.(10) and (6) become

$$\left[i\omega M_j - i \frac{M_j \omega^2}{\omega} + \sum_n \frac{V_{nj}}{V_{mj}} z_{nmj} + \sum_i \sum_n \frac{V_{ni}}{V_{mj}} Z_{nmi} \right] V_{mj} = n_{mj} E_j - f_{mj} , \quad (18)$$

$$I_j = \sum_n V_{nj} n_{nj} + Y_{bj} E_j . \quad (19)$$

Eqs.(18) and (19) are the main results of the general formulation. From them the transducer and array behaviour can be determined without knowing the velocity distributions of the transducers. It is necessary, of course, to determine the normal mode functions, the transfer impedance function, and the electromechanical transfer function and to solve the associated acoustics problem. To make these equations look more familiar we can rewrite them for the case of one transducer vibrating in one mode where we drop the i, j subscripts and let $n = m = 0$:

$$f_o = -Z_M V_o + n_o E \quad (18a)$$

$$I = n_o V_o + Y_b E \quad (19a)$$

where

$$Z_M = i\omega M - i \frac{M\omega_o^2}{\omega} + Z_{oo} + Z_{os}$$

is the total mechanical impedance of the transducer.

The sound field produced by the array at any point \vec{R} in the medium is

$$\begin{aligned} p(\vec{R}) &= \sum_i \iint v(\vec{r}_i) G(\vec{R}, \vec{r}_i) dS_i \\ &= \sum_i \sum_n v_{ni} \iint \eta_n(\vec{r}_i) G(\vec{R}, \vec{r}_i) dS_i = \sum_i \sum_n v_{ni} p_{ni}(\vec{R}). \end{aligned} \quad (20)$$

In this linear superposition note that the dependence on how the transducers are driven enters only through the v_{ni} , since the p_{ni} are independent of the driving.

The total time average acoustic power radiated by the array can be obtained by integrating the product of the pressure and the normal velocity over the entire surface of the array:

$$P = \frac{1}{2} \operatorname{Re} \iint v^* p dS.$$

For transducers mounted in a rigid baffle this reduces to the sum of integrals over the individual radiating surfaces:

$$\begin{aligned}
 P &= \frac{1}{2} \operatorname{Re} \sum_j \iint \dot{r}^* (\vec{r}_j) p(\vec{r}_j) dS_j \\
 &= \frac{1}{2} \operatorname{Re} \sum_j \sum_m \sum_i \sum_n \dot{v}_{mj}^* \dot{v}_{ni} Z_{nmij}
 \end{aligned} \tag{21}$$

The quantity

$$P_j = \frac{1}{2} \operatorname{Re} \sum_m \sum_i \sum_n \dot{v}_{mj}^* \dot{v}_{ni} Z_{nmij}$$

is the power radiated by the jth transducer when all the others are vibrating. It can also be calculated from the total radiation impedance of each mode,

$$Z_{mj} = \sum_n \sum_i \frac{\dot{v}_{ni}}{\dot{v}_{mj}} Z_{nmij}.$$

The power

$$P_{mj} = \frac{1}{2} \dot{v}_{mj} \dot{v}_{mj}^* \operatorname{Re} Z_{mj} = \frac{1}{2} \operatorname{Re} \sum_n \sum_i \dot{v}_{ni} \dot{v}_{mj}^* Z_{nmij}$$

is associated with the mth mode of the jth transducer including the coupling of the mth mode to all the other modes of all the transducers.

Thus the total power radiated by the jth transducer is

$$\sum_m P_{mj} = P_j.$$

Note that Z_{nmij} and Z_{mj} are the useful radiation impedances, and that they are defined with reference to the mode velocities.

Summary of General Procedure

The mathematical model consists of Eqs.(18) and (19) with the

preceding equations from which quantities such as Z_{nmij} can be calculated. The following steps are involved in using these equations:

- 1) Treat the radiating surfaces of the transducers as elastic bodies and determine appropriate normal mode functions, $\eta_m(\vec{r}_j)$, and frequencies, ω_{mj} , from Eq.(7) and the boundary conditions at the edges of the radiating surface.
- 2) Determine the acoustic pressure functions, p_{ni} , in Eq.(11) for the baffle and array geometry in question and calculate the mutual radiation impedances, Z_{nmij} , in Eq.(12). For a receiving problem calculate the f_{mj} in Eq.(17).
- 3) From the transducer structure determine a transfer impedance function, $z(\vec{r}_j, \vec{r}'_j)$, and an electromechanical transfer function, $n(\vec{r}_j)$, and calculate the z_{nmj} in Eq.(15) and the n_{mj} in Eq.(16).
- 4a) For transmitting put these quantities in Eq.(18), limit to a finite number of modes, specify the transducer voltages, E_j , and solve for the mode velocities, V_{mj} .
- 4b) For receiving specify the electrical termination (for example, open circuit with the $I_j = 0$), use Eq.(19) to eliminate the E_j from Eq.(18), and solve for the V_{mj} .
- 5) The velocity distribution can now be calculated from Eq.(9).
- 6) The sound pressure in the near or far field can be found from Eq.(20) and the radiated power from Eq.(21).

7) Eq.(19) gives the transducer currents for transmitting or the transducer voltages for receiving. It also gives the electrical admittance which is often needed in analyzing measurements.

The first three steps show how the analysis has been divided into three parts (elastic, acoustic, transducer structure and mechanism) which can be studied separately. The influence of the three parts on each other enters in the fourth step where the mode velocities are calculated. The remaining steps determine particular aspects of the transducer and array behaviour.

In the second part of this report we will discuss particular examples of the three parts of the problem. We will also consider the simple case of one transducer and two modes and illustrate solving Eq.(18) and interpreting the results. These examples are intended to clarify the steps listed above and to help make the general model readily usable in practical problems.

Examples of Use of the Model

Example of an Array

As a particular example consider a spherical array of identical transducers with flat, rectangular radiating surfaces as illustrated in Fig. 1. The acoustic part of the analysis of such an array has been formulated for the case where the individual radiating surfaces are much smaller than the radius of the sphere and the spherical baffle is rigid.⁹

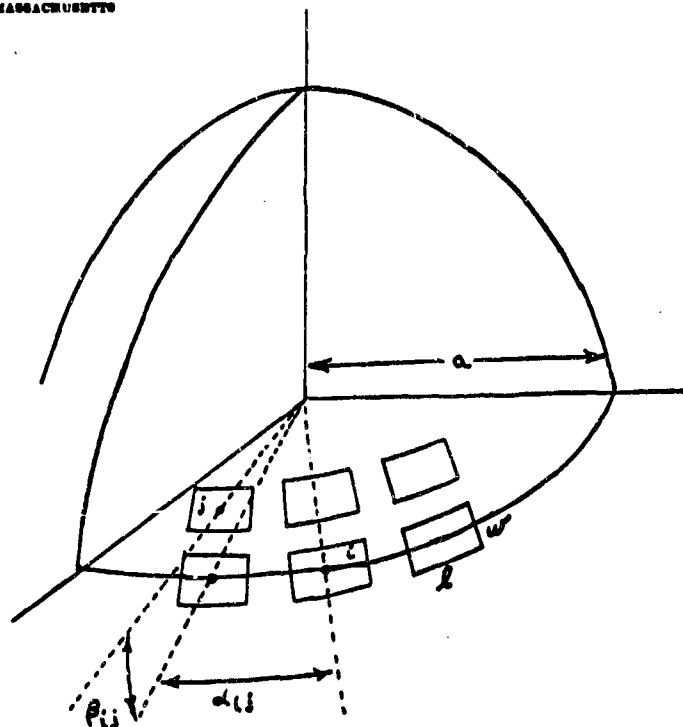


Figure 1. Spherical array of transducers with rectangular heads

It is then a good approximation to replace the flat radiating surfaces by the slightly curved portions of a rigid spherical surface. With this simplification the modal pressure functions can be found by the classical method of expansion in orthogonal functions. The result is

$$p_{ni}(\vec{R}_i) = p_{ni}(R, \theta_i, \varphi_i) = i \rho_w c_w \sum_{\mu=0}^{\infty} \sum_{\nu=-\mu}^{\mu} U_{\mu\nu ni} \frac{h_{\mu}(kR)}{h'_{\mu}(ka)} Y_{\mu}^{\nu}(\theta_i, \varphi_i) \quad (22)$$

where

$$U_{\mu\nu ni} = \int_{-\varphi_0}^{\varphi_0} \int_{\pi/2 - \theta_0}^{\pi/2 + \theta_0} \eta_n(\theta_i, \varphi_i) Y_{\mu}^{\nu*}(\theta'_i, \varphi'_i) \sin \theta'_i d\theta'_i d\varphi'_i,$$

h_μ is the spherical Hankel function, h'_μ is its derivative, Y_μ^ν is the normalized spherical harmonic, a is the radius of the sphere, ρ_w is the density of the medium, c_w is the speed of sound and $k = \omega/c_w$. In Eq.(12) \vec{R}_i is referred to a spherical coordinate system with origin at the center of the array and oriented so that the point $R = a$, $\theta_i = \pi/2$, $\varphi_i = 0$ is at the center of the rectangular head of the i th transducer. The normal modes of the head, which would naturally be expressed in rectangular coordinates (x_i, y_i) with origin at the center of the head, can be expressed in this coordinate system by the transformation

$$\begin{aligned}x_i &= a \varphi_i, \\y_i &= a (\pi/2 - \theta_i).\end{aligned}$$

The angles θ_0 and φ_0 are related to the length, l , and width, w , of the head by

$$\begin{aligned}\theta_0 &= w/2a, \\ \varphi_0 &= l/2a.\end{aligned}$$

To calculate the mutual radiation impedances we use Eq.(22) in Eq.(12) and obtain

$$Z_{nmij} = i \rho_w c_w a^2 \int_{-\varphi_0}^{\varphi_0} \int_{\pi/2-\theta_0}^{\pi/2+\theta_0} \sum_{\mu=0}^{\infty} \sum_{\nu=-\mu}^{\mu} U_{\mu\nu ni} \frac{h_\mu(ka)}{h'_\mu(ka)} Y_\mu^\nu(\theta_i, \varphi_i) Y_\mu^\nu(\theta_j, \varphi_j) \sin \theta_j d\theta_j d\varphi_j \quad (23)$$

It is necessary to express $Y_\mu^\nu(\theta_i, \varphi_i)$ in terms of the angles θ_j and φ_j of the coordinate system for the j th head which is rotated with respect to the coordinate system for the i th head. The transformation required can be written

$$Y_{\mu}^{\nu}(\theta_i, \varphi_i) = \sum_{\nu'=\mu}^{\infty} D_{\nu'\nu}^{\mu}(\alpha_{ij}, \beta_{ij}, \gamma_{ij}) Y_{\mu}^{\nu'}(\theta_j, \varphi_j) \quad (24)$$

where the general expression for $D_{\nu'\nu}^{\mu}$ is given in reference 9. In practical arrays the transducers would usually be oriented with the sides of their radiating surfaces parallel. In that case $\gamma_{ij} = 0$ and

$$D_{\nu'\nu}^{\mu}(\alpha_{ij}, \beta_{ij}, 0) = e^{-i\nu'\alpha_{ij}} d_{\nu'\nu}^{\mu}(\beta_{ij})$$

$$\text{where } d_{\nu'\nu}^{\mu}(\beta_{ij}) = \left[\frac{(\mu-\nu)!(\mu+\nu)!}{(\mu+\nu)!(\mu-\nu)!} \right]^{1/2} \frac{(\cos \frac{1}{2} \beta_{ij})^{2\mu+\nu-\nu'} (-\sin \frac{1}{2} \beta_{ij})^{\nu'-\nu}}{(\nu'-\nu)!}$$

$$\times F(\nu'-\mu, -\nu-\mu; \nu'-\nu+1; -\tan^2 \frac{1}{2} \beta_{ij}), \quad \nu' \geq \nu,$$

$$d_{\nu'\nu}^{\mu} = (-)^{\nu'-\nu} d_{\nu\nu'}^{\mu}, \quad \nu' < \nu,$$

F is the hypergeometric function (a finite polynomial), α_{ij} is the angular difference in azimuth and β_{ij} is the angular difference in latitude (see Fig. 1). Using Eq.(24) in Eq.(23) we find

$$Z_{nmij} = i \rho_w c_w a^3 \sum_{\mu=0}^{\infty} \sum_{\nu=-\mu}^{\mu} U_{\mu\nu ni} \frac{h_{\mu}(ka)}{h'_{\mu}(ka)} \sum_{\nu'=\mu}^{\infty} D_{\nu'\nu}^{\mu}(\alpha_{ij}, \beta_{ij}, 0) U_{\mu\nu' mj}^* \quad (25)$$

Eq.(25) gives all the acoustical quantities required for solving Eq.(18), while Eq.(22) gives the modal pressure functions needed in Eq.(20). Since these series are complicated and slowly converging for

$ka \gg 1$, their transformation to rapidly converging residue series¹⁰ or a more numerical approach¹¹ should be considered.

There has been little done on the many specialized radiation impedance problems which are involved in considering non-uniform velocity distributions. Mangulis¹² discussed the rocking mode of a rigid circular disk in an infinite, rigid plane and compared the results to the familiar piston mode. He defined the radiation impedance of the rocking mode in terms of moments, but the result is consistent with our general definition in Eq.(12). In our notation, with $n=0$ representing the piston mode and $n=1$ the rocking mode, Mangulis included the impedances Z_{00ij} and Z_{11ij} . He did not discuss $Z_{01ij} = Z_{10ij}$, but Eq.(14) shows that these vanish.

Another example is Porter's¹³ calculation of the self and mutual radiation impedances of flexural circular disks in an infinite rigid plane. In this case the quantities under discussion are Z_{00ij} and Z_{22ij} with $n=0$ representing the piston mode and $n=2$ the first axisymmetric bending mode. Approximate expressions were used in these calculations for the bending normal mode functions for supported and clamped edges. These boundary conditions prevent the piston mode from accompanying the bending mode; so that the coupling impedances Z_{02ij} and Z_{20ij} are not required.

Example of a Transducer

The longitudinal resonator transducer driven by a piezoceramic ring with a concentric tie rod is a good case for illustrating the transfer

impedance function, $z(\vec{r}_j, \vec{r}_j')$, and the electromechanical transfer function, $n(\vec{r}_j)$. Such a transducer usually has waterproof seals between the head and the housing, but since measurements show that these seals have little effect on the vibration, except to add damping, we will neglect them here. We assume that the tie rod contacts the inside of the radiating surface at a point which we take as the origin of coordinates. We also assume that the ceramic ring is thin enough to consider that it contacts the radiating surface in a circle of radius r_0 about the origin. We do not have to specify the type of head, its shape or its boundary conditions for this part of the problem.

When the surface moves the tie rod exerts a force on it at the point where the two are in contact. This force is proportional to the velocity of the surface at that point. We can write the part of $z(\vec{r}_j, \vec{r}_j')$ contributed by the tie rod as

$$Z_v \delta(\vec{r}_j' - \vec{r}_j) \delta(\vec{r}_j),$$

where Z_v is the mechanical impedance at the end of the tie rod attached to the head.

The ceramic ring exerts a force at all points of the circle where it contacts the head. One part of this force is electromechanical, and the corresponding transfer function can be written

$$n(\vec{r}_j) = \frac{N}{2\pi r_0} \delta(r_j - r_0), \quad (26)$$

where r_j is the radial polar coordinate and N is the electromechanical transfer ratio. The other part of the force exerted by the ceramic at

any point of the circle depends on the velocity of the head at all other points of the circle, because of elastic coupling through the ceramic. A simple approximation for the corresponding part of $\mathcal{J}(\vec{r}_j, \vec{r}_j')$ is to assume that the force exerted on the surface at \vec{r}_j is proportional to the velocity at \vec{r}_j . In this locally reacting approximation we denote the contribution to $\mathcal{J}(\vec{r}_j, \vec{r}_j')$ by

$$Z_c \delta(\vec{r}_j' - \vec{r}_j) \frac{i}{2\pi r_0} \delta(r_j - r_0)$$

where Z_c is the mechanical impedance at the end of the ceramic attached to the head.

The locally reacting approximation is similar to considering the ceramic ring divided lengthwise into many separate thin rods with no elastic coupling between them. It is not clear how adequate this approximation is, but it is very convenient, because it avoids treating another elastic problem associated with head flexing. The motion in the ceramic is usually treated as a one dimensional wave depending on position along the length of the ceramic. But when the head bends the motion in the ceramic also depends on position around the ring except for circular heads undergoing axially symmetric bending.

It is consistent with the locally reacting approximation to use expressions of the form¹⁴

$$Z = \rho_m c_m A \frac{Z_L + i \rho_m c_m A \tan k_m L}{\rho_m c_m A + i Z_L \tan k_m L}$$

for Z_c and Z_t . In this way other structural features of the transducer

such as a segmented ceramic stack, a tail mass and pressure release material can be included in the analysis. In this expression ρ_m is the density, c_m is the longitudinal speed of sound, A is the cross sectional area, and L is the length of the ceramic or tie rod; Z_L is the mechanical impedance at the end opposite the head, and $k_m = \omega/c_m$.

Combining these expressions gives

$$f(\vec{r}_j, \vec{r}_j') = Z_e \delta(\vec{r}_j' - \vec{r}_j) \delta(\vec{r}_j) + \frac{1}{2\pi r_0} Z_c \delta(\vec{r}_j' - \vec{r}_j) \delta(r_j - r_0). \quad (27)$$

Using this result in Eq.(15) we find

$$f_{nmj} = Z_e \eta_n(0) \eta_m(0) + Z_c b_{nmj}, \quad (28)$$

where

$$b_{nmj} = b_{mnj} = \frac{1}{2\pi r_0} \iint \eta_n(\vec{r}_j) \eta_m(\vec{r}_j) \delta(r_j - r_0) dS_j = \frac{1}{2\pi} \int_0^{2\pi} \eta_n(r_0, \theta) \eta_m(r_0, \theta) d\theta. \quad (29)$$

The last step, in which one integration is done in polar coordinates, holds for heads of any shape as long as they entirely cover the end of the ceramic ring. Similarly, using Eq.(26) in Eq.(16) gives

$$\eta_{mj} = \frac{N}{2\pi} \int_0^{2\pi} \eta_m(r_0, \theta) d\theta. \quad (30)$$

Eqs(28)-(30) give all the transducer parameters which are needed in Eq.(18).

Example of Normal Mode Functions

Metal plates are often used for the radiating surfaces of underwater

sound transducers. For circular plates the three familiar boundary conditions have all been treated to some extent (clamped,¹⁵ supported¹⁶ and free¹⁷). For rectangular plates only the supported edge case has been solved exactly,¹⁸ and it is sometimes used as an approximate model for plates fastened in such a way that the edge displacement is zero but the other conditions are not known. The rectangular plate with free edges is an important case for transducer applications, but the normal mode functions must be determined by approximate methods.¹⁹

Another boundary condition, different from the familiar three and leading to a simple, exact solution for rectangular plates, consists of vanishing of the first and third derivatives of the displacement in the direction normal to the edge. This condition might serve as an approximate model for plates with edges which can move, but are not completely free of forces and moments. However, it is quite different from the true free edge condition.

Although free edge rectangular plates do not have truly one dimensional modes,²⁰ they do have nearly one dimensional modes when one side is much longer than the other. The known normal modes for the bending vibrations of bars with free ends could be used as an approximation for the latter case. We will take these bar modes as a specific example which is simple enough to be discussed here. We thus imagine the type of transducer discussed in the previous section having a rectangular head with $w \ll l$ and also remember that $r_o \ll w$ for the ceramic ring to fit on the head.

For a thin bar in which shear and rotatory inertia can be neglected the appropriate differential operator is given by Eq.(2c) where x denotes position along the length of the bar. Using this operator in Eq.(7) and assuming that the bar is uniform and has a rectangular cross section we have

$$\frac{Y_t^2}{12\rho} \frac{d^4}{dx^4} \eta_n(x) = \omega_n^2 \eta_n(x) . \quad (31)$$

Taking the origin at the center of the bar the boundary conditions for free ends are

$$\frac{d^2 \eta_n}{dx^2} = \frac{d^3 \eta_n}{dx^3} = 0 , \quad x = \pm l/2 . \quad (32)$$

The normalization condition in Eq.(8) reduces to

$$\int_{-l/2}^{l/2} \eta_n(x) \eta_m(x) dx = l \delta_{nm} . \quad (33)$$

Meirovitch's²¹ discussion of the normal modes of the free bar is one of the most useful, because it includes the two zero frequency modes representing rigid body transverse translation and rigid body rotation about the center (note that Meirovitch takes the origin at one end of the bar). Any linear combination of these two modes is also a normal mode. The solutions of Eqs.(31) and (32) also include an infinite sequence of bending modes with increasing natural frequencies.

The normalized normal mode functions for the two rigid body motions

are:

$$\eta_0 = \omega_0 = 0 \quad (34)$$

$$\eta_n = \frac{\pi}{n} \quad \omega_n = 0 \quad (35)$$

... (ist. de) is sym. With respect to the center,
... (ing mode) is antisymmetric.

The bending modes can be divided into two groups; the symmetric ones

are:

$$\eta_n(x) = a_n \left[\cosh k_n x + \frac{\cosh \frac{k_n l}{2}}{\cos \frac{k_n l}{2}} \cos k_n x \right], \quad n = 2, 4, \dots \quad (36)$$

where k_n is a root of

$$\tanh \frac{k_n l}{2} + \tan \frac{k_n l}{2} = 0, \quad (37)$$

and

$$a_n^2 = \frac{2 \cos^2 \frac{k_n l}{2}}{\cosh^2 \frac{k_n l}{2} + \cos^2 \frac{k_n l}{2}}, \quad n = 2, 4, \dots \quad (38)$$

The antisymmetric bending modes are

$$\eta_n(x) = a_n \left[\sinh k_n x - \frac{\sinh \frac{k_n l}{2}}{\sin \frac{k_n l}{2}} \sin k_n x \right], \quad n = 3, 5, \dots \quad (39)$$

where k_n is a root of

$$\tanh \frac{k_n l}{2} - \tan \frac{k_n l}{2} = 0 \quad (40)$$

and

$$a_n^2 = \frac{2 \sin^2 \frac{k_n l}{2}}{\sinh^2 \frac{k_n l}{2} - \sin^2 \frac{k_n l}{2}}, \quad n = 3, 5 \dots \quad (41)$$

The natural frequencies are related to the k_n by

$$\omega_n^2 = \gamma t^2 k_n^4 / 12 \rho, \quad (42)$$

and the values of k_n are given by

$$\begin{aligned} k_2 l &= 1.506 \pi \\ k_3 l &= 2.500 \pi \\ k_n l &= (n - 1/2) \pi, \quad n = 4, 5 \dots \end{aligned}$$

Since the hyperbolic functions greatly exceed the trigonometric functions, the expressions for a_n simplify and the exact normal mode functions above can be written approximately as

$$\eta_n(x) \approx \sqrt{2} \left[\frac{\cos \frac{k_n l}{2}}{\cosh \frac{k_n l}{2}} \cosh k_n x + \cos k_n x \right], \quad n = 2, 4 \dots \quad (36a)$$

$$\eta_n(x) \approx \sqrt{2} \left[\frac{\sin \frac{k_n l}{2}}{\sinh \frac{k_n l}{2}} \sinh k_n x - \sin k_n x \right], \quad n = 3, 5 \dots \quad (39a)$$

Since the first normal mode is a constant it follows from the orthogonality condition that

$$\int_{-l/2}^{l/2} \eta_m(x) dx = 0, \quad m > 0. \quad (43)$$

This means that the volume velocity of all the rocking and bending modes is zero, and these modes are relatively poor acoustic radiators when $l \lesssim \lambda$ where λ is the acoustic wavelength.

Evaluation of Modal Velocities

We will illustrate solving Eq.(18) for the modal velocities by considering one transducer. Then we can omit the i, j subscripts, and Eq.(18) becomes

$$\left[i\omega M - i \frac{M\omega_m^2}{\omega} + \sum_n \frac{V_n}{V_m} (Z_{nm} + Z_{nm}) \right] V_m = n_m E. \quad (44)$$

We consider the transducer to consist of a tie rod and ceramic ring attached at the center of a rectangular head which can be described by the bar normal mode functions. Since the transducer structure is symmetric only symmetric normal modes will be excited by an applied voltage. Coupling of modes by the transducer structure and by the sound field will also excite only symmetric modes. Using Eq.(34) for the piston mode in Eqs.(29) and (30) we find

$$b_{00} = 1, \\ n_0 = N.$$

Now using Eq.(36) for the symmetric bending modes in Eq.(30) gives

$$n_m = N \frac{a_m}{2\pi} \int_0^{2\pi} \left[\cosh(k_m r_0 \cos \theta) + \frac{\cosh \frac{k_m l}{2}}{\cos \frac{k_m l}{2}} \cos(k_m r_0 \cos \theta) \right] d\theta \\ = N a_m \left[I_0(k_m r_0) + \frac{\cosh \frac{k_m l}{2}}{\cos \frac{k_m l}{2}} J_0(k_m r_0) \right], \quad m = 2, 4, 6 \dots$$

where J_0 is the zero order Bessel function, and I_0 is the zero order modified Bessel function. When Eq.(36) is used in Eq.(29) the integrations can be carried out, but the complicated formula that results probably offers no advantage over direct numerical integration. Note that for these normal mode functions

$$b_{0m} = n_m/N ,$$

because $\eta_0 = 1$.

Table 1 summarizes some of the numerical results which are required for solving Eq.(18) for one value of the ratio r_0/l .

Table 1. Numerical parameters for $r_0/l = .212$

n	$k_n l$	$\omega_n l (\rho/\gamma k)^{1/2}$	a_n	$\eta_n(0)$	n_n/N	$m=0$	b_{nm} $m=2$	$m=4$
0	0	0	1	1	1	1	-.833	.089
2	1.506π	$2.28\pi^2$.184	1.21	-.833	-.833	.773	-
4	2.500π	$6.25\pi^2$.0082	-.28	.089	.089	-	-

Using the results in Table 1 in Eq.(28) we have

$$z_{00} = Z_0 + Z_c ,$$

$$z_{22} = 1.46 Z_0 + .773 Z_c ,$$

$$z_{02} = z_{20} = -1.21 Z_0 - .833 Z_c .$$

The fact that n_4 is much smaller than n_0 or n_2 is understandable in terms of the zeros of the modes and the relative size of the ceramic ring and the head. The $n=0$ mode has no zeros, while the $n=2$ mode

has zeros at $\chi = \pm .276 l$. Since the ceramic ring, with radius $r_0 = .212 l$, lies between the zeros of the $n=2$ mode the driving force exerted by the ceramic is entirely in phase with this mode, and it, as well as the $n=0$ mode, is strongly excited. The $n=4$, and higher order, modes have more zeros, the ceramic ring extends over out of phase regions of the head, and these modes are not strongly excited.

Thus it will be a reasonable approximation to consider that only two modes exist. Then Eq.(44) gives two equations which are, using $\omega_0 = 0$,

$$(i\omega M + \gamma_{00} + Z_{00})V_0 + (\gamma_{20} + Z_{20})V_2 = n_0 E \quad (45)$$

$$(\gamma_{02} + Z_{02})V_0 + \left(i\omega M - i \frac{M\omega_2^2}{\omega} + \gamma_{22} + Z_{22}\right)V_2 = n_2 E \quad (46)$$

where $\gamma_{20} = \gamma_{02}$ and $Z_{20} = Z_{02}$. This two mode approximation is similar to the two part disk used by Woollett and Powers²² to simulate a flexing transducer head since both are two degree of freedom approximations to infinite degree of freedom systems. To emphasize the similarity and the differences the circuit diagram corresponding to Eqs.(45) and (46) is shown in Fig. 2.

Solving Eqs.(45) and (46) for V_0 and V_2 gives

$$V_0 = E \frac{n_0 \left(i\omega M - i \frac{M\omega_2^2}{\omega} + \gamma_{22} + Z_{22}\right) - n_2 (\gamma_{02} + Z_{02})}{(i\omega M + \gamma_{00} + Z_{00}) \left(i\omega M - i \frac{M\omega_2^2}{\omega} + \gamma_{22} + Z_{22}\right) - (\gamma_{02} + Z_{02})^2}, \quad (47)$$

$$V_2 = E \frac{n_2(i\omega M + \gamma_{00} + Z_{00}) - n_0(\gamma_{02} + Z_{02})}{(i\omega M + \gamma_{00} + Z_{00})(i\omega M - i\frac{M\omega^2}{\omega^2} + \gamma_{22} + Z_{22}) - (\gamma_{02} + Z_{02})^2} \quad (48)$$

These are the modal velocities that are produced by driving the transducer with the voltage E at the frequency ω . All aspects of the transducer behaviour can be derived from these velocities.

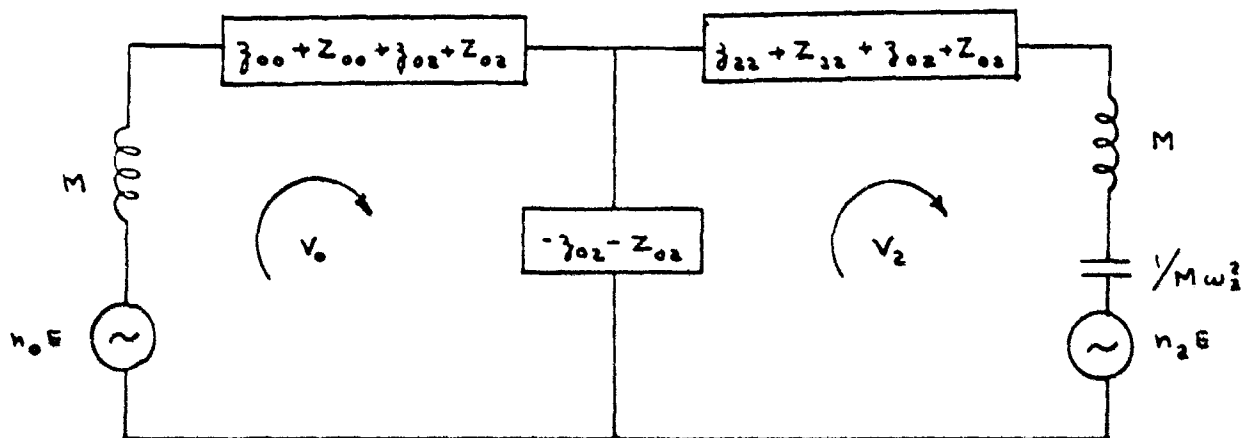


Figure 2. Circuit diagram corresponding to Eqs.(45) and (46) for a transducer with a flexing head.

This two mode approximation gives the first two normal modes of the transducer by setting $E = 0$. The two mode frequencies are the solutions of

$$(i\omega M + \gamma_{00} + Z_{00})(i\omega M - i \frac{M\omega_2^2}{\omega} + \gamma_{22} + Z_{22}) - (\gamma_{02} + Z_{02})^2 = 0, \quad (49)$$

obtained by equating the determinant of Eqs.(45) and (46) to zero. To solve this equation for ω it is necessary to specify the impedances Z_u and Z_c and the radiation impedances. As an approximation which holds for a heavy tail mass and a short ceramic stack and tie rod we take Z_u and Z_c to be pure compliances:

$$Z_u = -i/\omega C_t, \\ Z_c = -i/\omega C_c.$$

We also neglect the radiation impedances which holds for the transducer in air. The solutions of Eq.(49) can then be written

$$\omega^2 = \frac{1}{2M} \left\{ \left[M\omega_2^2 + \frac{\gamma_2^2(0)}{C_c} + \frac{b_{22}}{C_c} + \frac{\gamma_0^2(0)}{C_t} + \frac{b_{00}}{C_c} \right] \right. \quad (50)$$

$$\left. \pm \left[M\omega_2^2 + \frac{\gamma_2^2(0)}{C_c} + \frac{b_{22}}{C_c} - \frac{\gamma_0^2(0)}{C_c} - \frac{b_{00}}{C_c} \right] \left[1 + 4 \left(\frac{\frac{b_{02}}{C_c} + \frac{\gamma_2(0)}{C_c}}{M\omega_2^2 + \frac{\gamma_2^2(0)}{C_c} + \frac{b_{22}}{C_c} - \frac{\gamma_0^2(0)}{C_c} - \frac{b_{00}}{C_c}} \right)^2 \right]^{1/2} \right\}.$$

Expanding the radical gives

$$\omega_0^2 = \frac{1}{M} \left[\frac{\gamma_0^2(0)}{C_t} + \frac{b_{00}}{C_c} \right] - \frac{\left(\frac{b_{02}}{C_c} + \frac{\gamma_2(0)}{C_c} \right)^2}{M \left[M\omega_2^2 + \frac{\gamma_2^2(0)}{C_c} + \frac{b_{22}}{C_c} - \frac{\gamma_0^2(0)}{C_c} - \frac{b_{00}}{C_c} \right]} + \dots \quad (51)$$

and

$$\omega_2'^2 = \omega_2^2 + \frac{1}{M} \left[\frac{\eta_2^2(0)}{C_t} + \frac{b_{22}}{C_c} \right] + \frac{\left(\frac{b_{02}}{C_c} + \frac{\eta_2(0)}{C_t} \right)^2}{M \left[M\omega_2^2 + \frac{\eta_2^2(0)}{C_t} + \frac{b_{22}}{C_c} - \frac{\eta_0^2(0)}{C_t} - \frac{b_{00}}{C_c} \right]} + \dots \quad (52)$$

ω_0' is the longitudinal resonance of the transducer (resonance of the mass of the head with the compliance of the ceramic and tie rod). The first term of Eq.(51) is the usual expression for this resonance (since $\eta_0^2(0) = b_{00} = 1$), which would probably be used in a design neglecting head flexing. The other terms are modifications caused by head flexing since this mode is coupled to the bending mode. ω_2' is the resonance of the first symmetric bending mode of the transducer. Eq.(52) shows that this resonance, which is ω_2 for the head alone, is raised by the stiffening effect of the tie rod and ceramic (the second term), and further raised by coupling with the other mode (the additional terms).

The velocity distribution for these two modes of the transducer can be determined from the ratio

$$\frac{V_2}{V_0} = - \frac{i\omega M + \gamma_{00}}{\gamma_{02}} = - \frac{\gamma_{02}}{i\omega M - i \frac{M\omega_2^2}{\omega} + \gamma_{22}}, \quad (53)$$

evaluated at ω_0' and at ω_2' .

Discussion of the Example

We will not attempt a comprehensive discussion of the effects of head flexing in this report. However, to illustrate the use of the

results in the previous section we will mention some aspects of the transducer behaviour.

If we had analyzed the transducer on the assumption that the head was rigid we would have found the velocity of the piston mode to be

$$V_{or} = \frac{n_o \bar{E}}{i\omega M + \bar{z}_{oo} + Z_{oo}} \quad (54)$$

and, of course, there would be no other modal velocities such as V_2 .

Eq.(47) can be rewritten as

$$V_o = V_{or} \left[\frac{1 - \frac{n_2}{n_o} \frac{\bar{z}_{o2} + Z_{o2}}{(i\omega M - i M \omega_2^2 / \omega + \bar{z}_{22} + Z_{22})}}{1 - \frac{(\bar{z}_{o2} + Z_{o2})^2}{(i\omega M + \bar{z}_{oo} + Z_{oo})(i\omega M - i M \omega_2^2 / \omega + \bar{z}_{22} + Z_{22})}} \right] \quad (55)$$

showing that the factor in square brackets gives the effect of head flexing on the piston mode velocity.

The effects of head flexing on the directivity pattern and the radiated power are of great practical importance. Under the rigid head assumption the radiated power is

$$P_r = \frac{1}{2} |V_{or}|^2 R_{oo} \quad (56)$$

where R_{oo} is the radiation resistance associated with Z_{oo} . For the flexing head Eq.(21) gives

$$P = \frac{1}{2} |V_o|^2 R_{oo} + \frac{1}{2} |V_2|^2 R_{22} + \operatorname{Re}(V_o^* V_2) R_{o2} \quad (57)$$

$$= P_r \left[\left| \frac{V_o}{V_{or}} \right|^2 + \left| \frac{V_2}{V_{or}} \right|^2 \frac{R_{22}}{R_{oo}} + 2 \left| \frac{V_o}{V_{or}} \right| \left| \frac{V_2}{V_{or}} \right| \frac{R_{o2}}{R_{oo}} \cos \delta_{o2} \right]$$

where δ_{02} is the phase difference between V_0 and V_2 . Again the factor in square brackets shows the effect of head flexing. Quantitative evaluation of these effects obviously requires knowledge of the radiation impedances as a function of frequency which must be calculated numerically.

We will discuss the shape of the velocity distribution for the transducer in air where the radiation impedances can be neglected. Such calculations are required for interpreting holographic interferometry measurements which, so far, have only been made accurately in air. The velocity amplitudes $|V_{or}|$, $|V_0|$ and $|V_2|$ are given as a function of frequency for constant voltage in Fig. 3 where we have plotted the velocity amplitude divided by the constant quantity $C_e NE$. The computations made use of the numbers in Table 1 and the values

$$\begin{aligned} MC_e &= 0.167 \times 10^{-2}, \\ \omega_2^2 &= 10^9, \\ C_e/C_0 &= 0.1, \end{aligned}$$

all of which are typical of common types of Sonar transducers.

If the head was considered rigid there would be a velocity maximum at the frequency ω_{or} where the impedance $i\omega M + Z_{00}$ vanishes. This frequency corresponds to the first term in Eq.(51), that is

$$\omega_{or}^2 = \frac{1}{M} \left[\frac{Z_0(\omega)}{C_e} + \frac{b_{00}}{C_c} \right]. \quad (58)$$

Figure 3 shows how the single resonance ω_{or} for a rigid head is split into a higher and a lower resonance ω'_0 and ω'_2 when one bending mode is considered. It also shows how the two head mode frequencies ω_0 and ω_2 are raised to ω'_0 and ω'_2 when the head is attached to the rest of the transducer.

Figure 3 shows that for the flexing head at low frequency, V_0 is very similar to V_{or} and V_2 is very small. At the frequency ω'_0 V_0 and V_2 go through a maximum together, because the modes are coupled, and when the velocity of one mode is high it causes the other mode velocities to also be high. This is shown by Eqs.(47) and (48) for V_0 and V_2 which have the same denominator and thus the same maxima. Note that if $\beta_{02} + Z_{02} = 0$, corresponding to no coupling between modes, these equations would not have the same denominator. If resistance was included in the calculations a maximum in V_0 would still be accompanied by a maximum in V_2 , but at ω'_0 the maximum value of V_0 would exceed that of V_2 at least for small coupling ($\beta_{02} + Z_{02} \ll i\omega M - iM\omega_2^2/\omega + \beta_{22} + Z_{22}$) as can be seen from Eqs.(47) and (48). Similarly, both V_0 and V_2 have maxima at ω'_2 and with resistance the maximum value of V_2 would exceed that of V_0 .

V_0 goes to zero when the numerator of Eq.(47) vanishes, which occurs in this case at a frequency close to but not the same as ω_2 . The numerator of Eq.(48) shows that V_2 does not vanish in this frequency region. At high frequency V_0/V_2 approaches $n_0/n_2 = -1.2$. In Fig. 3 there is a phase change at each peak and at the zero of V_0 . Thus at

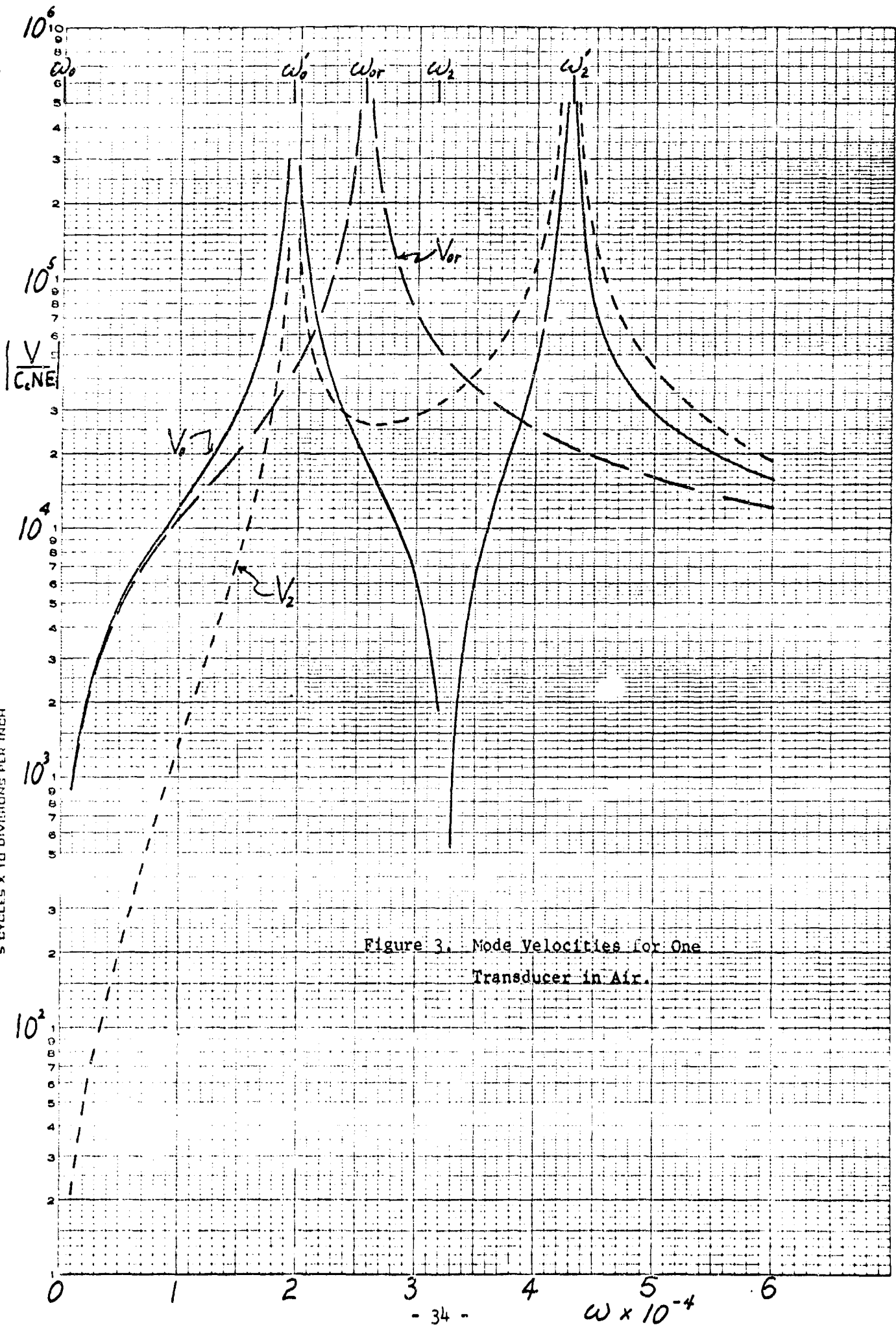


Figure 3. Mode Velocities for One Transducer in Air.

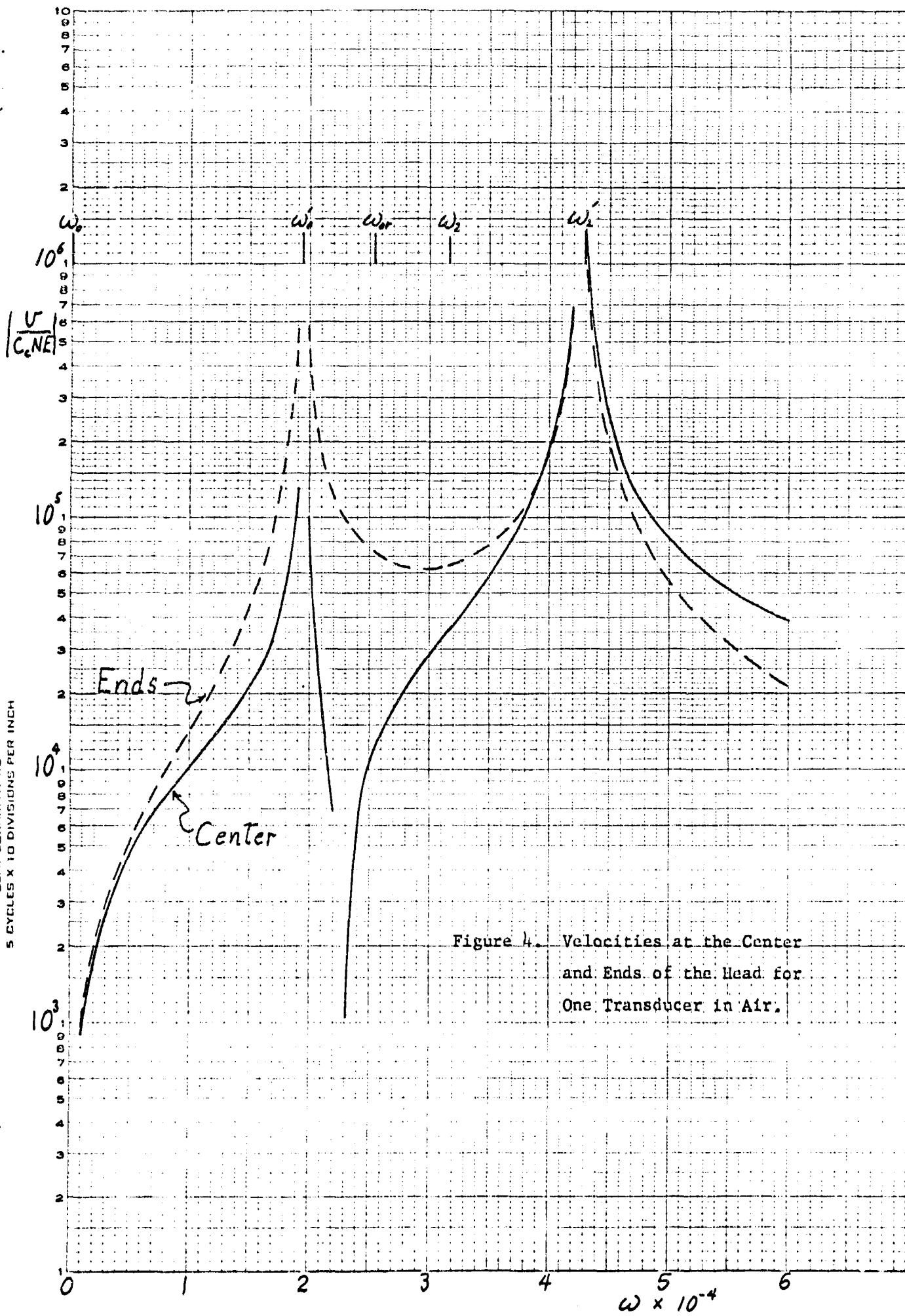


Figure 4. Velocities at the Center and Ends of the Head for One Transducer in Air.

high frequency V_0 and V_2 are out of phase.

The velocity distribution of the head is

$$v(x) = V_0 \eta_0(x) + V_2 \eta_2(x). \quad (59)$$

The velocities at the center, $v(0)$, and at the ends, $v(l/2)$, of the head are given as a function of frequency in Fig. 4. At very low frequency the motion of the head is almost uniform, but as the frequency comes closer to ω_0' bending increases and the velocity at the ends exceeds the velocity at the center by a factor of two or more. At a frequency slightly above ω_0' the velocity at the center goes to zero. As Fig. 3 shows this is the frequency where $V_0 = 1.21 V_2$ and the two modes cancel each other. Above this frequency the velocities of the ends and center are out of phase, and there is a node in the velocity distribution. In the vicinity of ω_2' the velocities of the ends and center have almost the same magnitude. At higher frequencies the velocity at the center exceeds that at the ends and the ratio $v(0)/v(l/2)$ approaches -3.

To show how the calculations can be related to electrical admittance measurements we note that Eq.(19) becomes in this case

$$I = V_0 n_0 + V_2 n_2 + Y_b E. \quad (60)$$

If we take $Y_b = i\omega C_b$, where C_b is the blocked capacitance, and use $Y = I/E$ we can rewrite Eq.(60) as

$$\frac{Y}{C_b N^2} = \frac{i\omega C_b}{C_b N^2} + \frac{n_0}{N} \frac{V_0}{C_b N E} + \frac{n_2}{N} \frac{V_2}{C_b N E}, \quad (61)$$

where n_0/N and n_2/N are given in Table 1 and the ratios $v_0/c_e N \epsilon$ and $v_2/c_e N \epsilon$ are given by Eqs.(47) and (48). The magnitudes of these ratios are shown in Fig. 2. Note also that

$$\frac{C_b}{C_e N^2} = \frac{1 - k^2}{k^2 (1 + C_e/C_b)}$$

where k is the electromechanical coupling factor. From Eq.(61) and the results in Fig. 2 it is clear that, when we neglect all resistance, we have large peaks superimposed on the blocked susceptance curve at ω'_0 and ω'_2 . If resistance was included in the calculations we would have admittance circles in the vicinity of these frequencies. In general we would have admittance circles at each of the coupled mode frequencies of the transducer.

Conclusion

A general mathematical model not restricted by the fixed velocity distribution assumption has been presented. It should be adequate for most transducer array problems, and it should be especially convenient for problems connected with transducer head flexing. The application of the general model to typical situations was discussed in order to further clarify how it can be used. This part of the discussion also focused attention on some of the more specialized types of problems which must be solved before general models can be used.

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13. ABSTRACT <p>A mathematical model for transducer arrays is developed which is not restricted to fixed velocity distribution transducers. The model is general enough to cover most transducers of interest for Sonar applications. Its formulation is also explicit and simple enough to make it readily usable in transducer and array design. It is especially well suited for analyzing the effects of transducer head flexing. The use of the model is illustrated by some discussion of a spherical array of longitudinal vibrator transducers with rectangular heads. Numerical calculations are given here for one transducer in air, and for this case some results related to head flexing are obtained and discussed.</p>		

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